North Sydney Boys High School

MATHEMATICS (EXTENSION 1)

2011 HSC Course Assessment Task 2

General instructions

- Working time – 60 minutes.
- Commence each new question on a new page.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M3A – Mr Lam
- ◯ 12M3B – Mr Weiss
- ◯ 12M3C – Mr Law
- ◯ 12M4A – Mr Fletcher/Mrs Collins
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mrs Collins/Mr Rezcallah

STUDENT NUMBER .... **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	4	5	Total	%
MARKS	/13	/12	/10	/9	/9	/53	

Question 1 (13 Marks) Commence a NEW page. **Marks**

(a) Differentiate each of the following with respect to x:

i. $x^3 e^{-2x}$ **2**

ii. $\dfrac{\ln x}{\cos x}$ **2**

iii. $\sin^2 3x$ **2**

iv. $\log_{10}(1-x)$ **2**

(b) Find:

i. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$ **2**

ii. $\displaystyle\int \frac{dx}{(3x-1)^2}$ **1**

iii. $\displaystyle\int \tan x\,dx$ **2**

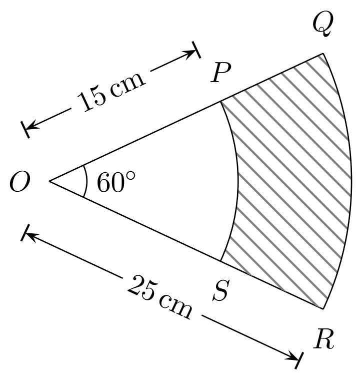
Question 2 (12 Marks) Commence a NEW page. **Marks**

(a) State the period and amplitude of $y = \dfrac{1}{3}\sin\left(2x - \dfrac{\pi}{4}\right)$. **2**

(b) Solve $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\theta$ for $0 \le \theta \le 2\pi$. **3**

(c) Evaluate $\displaystyle\lim_{x\to 0}\left(\frac{\sin 2x}{\tan 3x}\right)$. **2**

(d) PS and QR are arcs of concentric circles with centre O.
Calculate in terms of π,



i. the area of the shaded region $PQRS$. **2**

ii. The perimeter of the shaded region $PQRS$. **3**

Question 3 (10 Marks) Commence a NEW page. **Marks**

(a) Solve the equation $\ln(x+6) + \ln(x-3) = \ln 5 + \ln 2$ **3**

(b) i. Differentiate $x \cos x$. **1**

ii. Hence or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} x \sin x \, dx$. **3**

(c) Find the equation of the tangent to the curve $y = e^{\tan x}$ at the point on the curve where $x = \frac{\pi}{4}$. **3**

Question 4 (9 Marks) Commence a NEW page. **Marks**

(a) Solve the equation $e^{6x} - 7e^{3x} + 6 = 0$. **3**

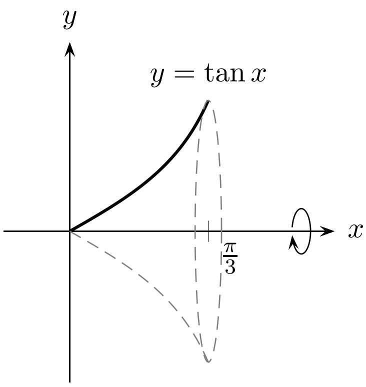
(b) Find $\dfrac{dy}{dx}$ if $y = \ln\left(\dfrac{x^4}{1-x^3}\right)$. **2**

(c) Use mathematical induction to show for $n \geq 1$, **4**

$$\frac{1}{1\times 5} + \frac{1}{5\times 9} + \frac{1}{9\times 13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$

Question 5 (9 Marks) Commence a NEW page. **Marks**

(a) The area between the curve $y = \tan x$, the x axis and the ordinates $x = 0$ and $x = \frac{\pi}{3}$ is rotated about the x axis. **3**



Find the exact volume of the solid which is generated.

(b) i. On the same set of axes, sketch the graphs of $y = \sin x$ and $y = 1 - \cos x$ for $0 \leq x \leq \pi$. **2**

ii. Find the values of x where $\sin x = 1 - \cos x$ for $0 \leq x \leq \pi$. **2**

iii. Calculate the area enclosed the curves $y = \sin x$ and $y = 1 - \cos x$ for $0 \leq x \leq \pi$. **2**

End of paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \ x > 0$

Suggested Solutions

Question 1

(a) i. (2 marks)

✓ [1] for correct application of product rule.

✓ [1] for final answer.

$$\frac{d}{dx}\left(x^3e^{-2x}\right)$$

$$u = x^3 \qquad v = e^{-2x}$$

$$u' = 3x^2 \qquad v' = -2e^{-2x}$$

$$\begin{aligned}\frac{d}{dx}\left(x^3e^{-2x}\right) &\\ &= x^3\cdot\left(-2e^{-2x}\right) + e^{-2x}\cdot 3x^2\\ &= \boxed{3x^2e^{-2x} - 2x^3e^{-2x}}\\ &= x^2e^{-2x}(3-2x)\end{aligned}$$

ii. (2 marks)

✓ [1] for correct application of quotient rule.

✓ [1] for final answer.

$$\frac{d}{dx}\left(\frac{\ln x}{\cos x}\right)$$

$$u = \ln x \qquad v = \cos x$$

$$u' = \frac{1}{x} \qquad v' = -\sin x$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\ln x}{\cos x}\right) &= \frac{\cos x\cdot\frac{1}{x} - \ln x\cdot\sin x}{\cos^2 x}\\ &= \boxed{\frac{\frac{\cos x}{x} + \sin x\ln x}{\cos^2 x}}\ {}^{\times x}_{\times x}\\ &= \frac{\cos x + x\sin x\ln x}{x\cos^2 x}\end{aligned}$$

iii. (2 marks)

✓ [1] for correct application of chain rule.

✓ [1] for final answer.

$$\begin{aligned}\frac{d}{dx}\left(\sin^2 3x\right) &= 2\sin 3x\times 3\cos 3x\\ &= \boxed{6\sin 3x\cos 3x}\end{aligned}$$

iv. (2 marks)

✓ [1] for correctly changing base from 10 to e.

✓ [1] for correct differentiation.

$$\begin{aligned}\frac{d}{dx}\left(\log_{10}(1-x)\right) &= \frac{d}{dx}\left(\frac{\ln(1-x)}{\ln 10}\right)\\ &= \boxed{\frac{1}{\ln 10}\times -\frac{1}{1-x}}\\ &= \frac{1}{\ln 10(x-1)}\end{aligned}$$

(b) i. (2 marks)

✓ [−1] for each mistake.

$$\int \frac{e^{2x}}{e^{2x}+1}\,dx = \boxed{\frac{1}{2}\ln\left(e^{2x}+1\right) + C}$$

ii. (1 mark)

$$\begin{aligned}\int \frac{dx}{(3x-1)^2} &= \int (3x-1)^{-2}\,dx\\ &= -\frac{1}{3}(3x-1)^{-1} + C\\ &= \boxed{-\frac{1}{3(3x-1)} + C}\end{aligned}$$

iii. (2 marks)

✓ [−1] for each mistake.

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx\\ &= -\int \frac{-\sin x}{\cos x}\,dx\\ &= \boxed{-\ln(\cos x) + C}\end{aligned}$$

Question 2

(a) (2 marks)

✓ [1] each for a and T.

$$y = \frac{1}{3}\sin\left(2x - \frac{\pi}{4}\right) \equiv a\sin(nx+\phi)$$

$$\boxed{a = \frac{1}{3}} \qquad \boxed{T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi}$$

(b) (3 marks)

✓ [1] for resolving $\tan\theta$ and $\operatorname{cosec}\theta$ into $\sin\theta$ and $\cos\theta$.

✓ [1] for each of $\frac{\pi}{6}$, $\frac{5\pi}{6}$.

$$\begin{aligned} 8\sin\theta\cos\theta\tan\theta &= \operatorname{cosec}\theta \\ 8\sin\theta \times \cancel{\cos\theta} \times \frac{\sin\theta}{\cancel{\cos\theta}} &= \frac{1}{\sin\theta} \\ 8\sin^3\theta &= 1 \\ \sin^3\theta &= \frac{1}{8} \\ \therefore \sin\theta &= \frac{1}{2} \\ \boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}} \end{aligned}$$

(c) (2 marks)

✓ [1] for manipulating original limit to become $\frac{2}{3} \times \frac{\sin 2x}{2x} \times \frac{3x}{\tan 3x}$. If this is not done, mark is not awarded.

✓ [1] for final answer.

$$\begin{aligned} \lim_{x\to 0}\left(\frac{\sin 2x}{\tan 3x}\right) &= \frac{2}{3} \times \lim_{x\to 0}\left(\frac{\sin 2x}{2x} \times \frac{3x}{\tan 3x}\right) \\ &= \frac{2}{3} \times 1 \times 1 = \boxed{\frac{2}{3}} \end{aligned}$$

(d) i. (2 marks)

$$\begin{aligned} A &= \frac{1}{2} \times 25^2 \times \frac{\pi}{3} - \frac{1}{2} \times 15^2 \times \frac{\pi}{3} \\ &= \boxed{\frac{200\pi}{3}\ \text{cm}^2} \end{aligned}$$

ii. (3 marks)

$$\begin{aligned} P &= PS + QR + PQ + RS \\ &= \left(15 \times \frac{\pi}{2}\right) + \left(25 \times \frac{\pi}{3}\right) + 10 + 10 \\ &= \boxed{20 + \frac{40\pi}{3}\ \text{cm}} \end{aligned}$$

Question 3

(a) (3 marks)

✓ [1] for $\ln\Big[(x+6)(x+3)\Big] = \ln(5\times 2)$.

✓ [1] for $x = -7, 4$.

✓ [1] for final answer.

✓ [0] for $\ln\left(\frac{x+6}{x-3}\right) = \ln 10$ as this sidesteps the quadratic and also the need to discard $x = -7$.

$$\begin{aligned} \ln(x+6) + \ln(x-3) &= \ln 5 + \ln 2 \\ \ln\Big[(x+6)(x+3)\Big] &= \ln(5\times 2) \\ (x+6)(x-3) &= 10 \\ x^3 + 3x - 28 &= 0 \\ (x+7)(x-4) &= 0 \\ x &= -7, 4 \end{aligned}$$

However, $\ln(-7+6)$ and $\ln(-7-3)$ are not defined.

$$\therefore \boxed{x = 4}$$

(b) i. (1 mark)

$$\begin{aligned} &\frac{d}{dx}(x\cos x) \\ u &= x \qquad v = \cos x \\ u' &= 1 \qquad v' = -\sin x \\ \therefore \frac{d}{dx}(x\cos x) &= -x\sin x + \cos x \end{aligned}$$

ii. (3 marks)

✓ [1] for $\int_0^{\frac{\pi}{3}} x\sin x\,dx = \int_0^{\frac{\pi}{3}} \cos x\,dx - \int_0^{\frac{\pi}{3}} \frac{d}{dx}(x\cos x)\ dx$

✓ [1] for finding the primitive of the two functions.

✓ [1] for final answer.

Since $\frac{d}{dx}(x\cos x) = -x\sin x + \cos x$,

$$\therefore x\sin x = \cos x - \frac{d}{dx}(x\cos x)$$

$$\begin{aligned} &\therefore \int_0^{\frac{\pi}{3}} x\sin x\ dx \\ &= \int_0^{\frac{\pi}{3}} \cos x\ dx - \int_0^{\frac{\pi}{3}} \frac{d}{dx}(x\cos x)\ dx \\ &= \Big[\sin x\Big]_0^{\frac{\pi}{3}} - \Big[x\cos x\Big]_0^{\frac{\pi}{3}} \\ &= \left(\frac{\sqrt{3}}{2} - 0\right) - \left(\frac{\pi}{3}\cdot\frac{1}{2} - 0\right) \\ &= \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{6}} \end{aligned}$$

(c) (3 marks)

$$y = e^{\tan x} \quad \Rightarrow \quad \frac{dy}{dx} = \sec^2 x e^{\tan x}$$

At $x = \frac{\pi}{4}$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cos^2 \frac{\pi}{4}} e^{\tan \frac{\pi}{4}} \\ &= \left(\sqrt{2}\right)^2 \times e^1 \\ &= 2e \\ y = e^{\tan \frac{\pi}{4}} &= e \end{aligned}$$

By the point gradient formula,

$$y - e = 2e\left(x - \frac{\pi}{4}\right)$$

$$\boxed{y = 2ex - \frac{e\pi}{2} + e}$$

Question 4

(a) (3 marks)

- ✓ [1] for correct factorisation.
- ✓ [1] for $e^{3x} = 1$ and $e^{3x} = 6$.
- ✓ [1] for final solutions.

$$e^{6x} - 7e^{3x} + 6 = 0$$

Let $m = e^{3x}$,

$$\begin{aligned} m^2 - 7m + 6 &= 0 \\ (m-6)(m-1) &= 0 \\ m &= 1, 6 \\ \therefore e^{3x} &= 1, 6 \end{aligned}$$

$e^{3x} = 1$	$e^{3x} = 6$
$3x = 0$	$3x = \ln 6$
$x = 0$	$x = \frac{1}{3}\ln 6$

$$\therefore \boxed{x = 0 \text{ or } \frac{1}{3}\ln 6}$$

(b) (2 marks)

- ✓ [1] for using log rules to resolve fraction into a simpler expression.
- ✓ [1] for final answer.

$$y = \ln\left(\frac{x^4}{1-x^3}\right) = \ln x^4 - \ln(1-x^3)$$

$$\frac{dy}{dx} = \frac{4}{x} - \frac{-x^2}{1-x^3} = \boxed{\frac{4}{x} + \frac{x^2}{1-x^3}}$$

(c) (4 marks)

- ✓ [1] for correct evaluation of base case.
- ✓ [1] for correct assumption in inductive step.
- ✓ [2] for correct inductive step part 2.

$$\frac{1}{1\times 5} + \frac{1}{5\times 9} + \frac{1}{9\times 13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$

- Base case: $n = 1$

$$\frac{1}{1\times 5} = \frac{1}{5}$$
$$\frac{1}{4(1)+1} = \frac{1}{5}$$

  Hence the statement is true for $n = 1$.

- Inductive step: assume the statement is true for some value $k \in \mathbb{R}$, i.e.

$$\frac{1}{1\times 5} + \frac{1}{5\times 9} + \frac{1}{9\times 13} + \cdots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

  and evaluate the statement when $n = k+1$:

$$\begin{aligned} &\overbrace{\frac{1}{1\times 5} + \cdots + \frac{1}{(4k-3)(4k+1)}}^{=\frac{k}{4k+1}} + \frac{1}{\big(4(k+1)-3\big)\big(4(k+1)+1\big)} \\ &= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)} \\ &= \frac{k(4k+5)+1}{(4k+1)(4k+5)} \\ &= \frac{4k^2+5k+1}{(4k+1)(4k+5)} \\ &= \frac{\cancel{(4k+1)}(k+1)}{\cancel{(4k+1)}(4k+5)} \\ &= \frac{k+1}{4(k+1)+1} \end{aligned}$$

The statement is also true for $n = k + 1$. By the principle of mathematical induction, the statement is true for all positive integers n.

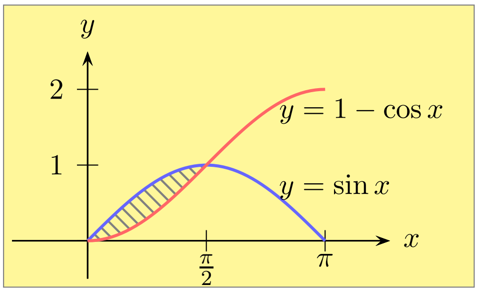
Question 5

(a) (3 marks)

$$V = \pi \int y^2\,dx = \pi \int_0^{\frac{\pi}{3}} \tan^2 x\,dx$$
$$= \pi \int_0^{\frac{\pi}{3}} \sec^2 x - 1\,dx$$
$$= \pi \Big[\tan x - x\Big]_0^{\frac{\pi}{3}} = \pi\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right)$$
$$= \boxed{\pi\left(\sqrt{3} - \frac{\pi}{3}\right)} = \pi\left(\frac{3\sqrt{3} - \pi}{3}\right)$$

(b) i. (2 marks)

- ✓ [1] for $y = 1 - \cos x$.
- ✓ [1] for $y = \sin x$.
- ✓ [−1] if *any* of the following are true:
  - no labels (one label = no penalty)
  - no scale
  - larger domain than required
  - extra solution(s) to x



ii. (2 marks)

$$\boxed{x = 0, \frac{\pi}{2}}$$

iii. (2 marks)

$$A = \int_0^{\frac{\pi}{2}} \sin x - (1 - \cos x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} \sin x + \cos x - 1\,dx$$
$$= \Big[-\cos x + \sin x - x\Big]_0^{\frac{\pi}{2}}$$
$$= \left(\sin\frac{\pi}{2} - \cancel{\sin 0}\right) - \left(\cancel{\cos\frac{\pi}{2}} - \cos 0\right) - \left(\frac{\pi}{2} - 0\right)$$
$$= \boxed{2 - \frac{\pi}{2}} = \frac{4 - \pi}{2}$$